ON CONTINUOUS EVASION STRATEGIES IN GAME PROBLEMS ON THE ENCOUNTER OF MOTIONS

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Relationships between certain types of absorption sets defined for the game problem of bringing a conflict-controlled motion onto a given set are considered. Investigation of these matters shows that an evasion strategy generally cannot be approximated by continuous game position vector functions. The contents of the present paper are directly related to studies [1−9].

1. One of the methods of investigating differential games is that described in [1, 2, 7]. In these studies construction of the optimal strategies of the players entails the introduction of an ancillary extremal construction bases on the notion of absorption. Absorption is defined in various ways, depending on the class of problems under consideration. We shall cite the classification of the various definitions of absorption proposed by Krasovskii; this classification will help us to define the subject of the present paper more clearly.

We shall limit ourselves to the consideration of a linear differential homing game. Let the motion of the controlled system be described by the equation

$$dx \,/\, dt = A\ (t)x + B\ (t)u - C\ (t)v \tag{1.1}$$

Here x is the k-dimensional phase vector of the system; u, v are the vector controls of the first and second players; $A\ (t)$, $B\ (t)$, $C\ (t)$ are continuous matrix functions of the corresponding dimensionalities. We assume that the realizations of the controlling forces of the players, $u\ [t]$ and $v\ [t]$, are subject to restrictions of the form

$$u\ [t] \in P, \qquad v\ [t] \in Q \tag{1.2}$$

where P, Q are closed, bounded, and convex sets in the corresponding vector spaces. In the phase space of the game E_k we are given some convex and closed set M to which the first player strives to bring the point $x\ [t]$; the second player, with the control v at his command, is not interested in the realization of the condition $x\ [t] \in M$. It is usually assumed that neither player knows what the future behavior of his opponent will be, but that the game position $p\ [t] = \{t,\ x\ \ [t]\}$ realized at each instant t immediately is known to both.

Let us define the basic classes of player strategies. This will enable us to introduce certain types of absorption sets into our discussion.

The first calss of strategies consists of the program controls at the disposal of the players, i.e. the arbitrary measurable vector functions $u\ (t)$ and $v\ (t)$ which satisfy restrictions (1.2). We denote the sets of program controls of the first and second players by the the symbols $\mathbf{U}_1$ and $\mathbf{V}_1$.

The second class of player strategies consists of the continuous vector functions $u = u\ (t,\ x)$, $v = v\ (t,\ x)$ which satisfy the conditions $u\ (t,\ x) \in P$, $v\ (t,\ x) \in Q$. We denote the sets of such vector functions by the symbols $\mathbf{U}_2$ and $\mathbf{V}_2$.

Let us consider another ancillary class of player strategies. Let $u = u\ (t,\ x)$,

$v = v(t, x)$ be vector functions satisfying the following conditions: the functions $u(t, x)$, $v(t, x)$ are measurable in t for each fixed x; there exists a function $L(t)$ summable over every finite interval such that the following inequalities (the Lipschitz condition) are fulfilled:

$$\| u(t, x_1) - u(t, x_2) \| \leqslant L(t) \| x_1 - x_2 \|$$

$$\| v(t, x_1) - v(t, x_2) \| \leqslant L(t) \| x_1 - x_2 \|$$

and the inclusions

$$u(t, x) \in P, \qquad v(t, x) \in Q \tag{1.3}$$

hold for all values $\{t, x\}$.

The sets of vector functions $u = u(t, x), v = v(t, x)$ satisfying these conditions are denoted by the symbols $\mathbf{U}_2^*$ and $\mathbf{V}_2^*$. By virtue of the Carátheodory theorem ([10], p. 120) the pair of controls $u(\cdot) \in \mathbf{U}_2^*$, $v(\cdot) \in \mathbf{V}_2^*$ generates a unique motion of system (1.1).

Now let us introduce the notion of an approximation strategy. Let $U = U(t, x)$ be a function which associates the vector $p = \{t, x\}$ with a closed set $U(t, x)$ which satisfies the inclusion $U(t, x) \subset P$. We also assume that the function $U = U(t, x)$ is semicontinuous above by inclusion with respect to the variable x. The function $U = U(t, x)$ enables us to describe the following method of forming the first players's controls, which is known as the "approximation strategy" [7].

Ler $\delta > 0$ be an arbitrary number (the interval of the approximation scheme); we choose an arbitrary vector u_0 in the set $U(t_0, x_0)$ which the function $U = U(t, x)$ associates with the initial game position $p_0 = \{t_0, x_0\}$; we then keep the control of the first player constant, $u[t] = u_0 \in U(t_0, x_0)$ within the half-interval $[t_0, t_0 + \delta)$. This constant control of the first player paired with the arbitrary measurable realization of the second player's control determines the motion of the system over the time interval $[t_0, t_0 + \delta)$. At the instant $t = t_0 + \delta$ we associate the game position $p = \{t_0 + \delta, x[t_0 + \delta]\}$ realized by that instant with the set $U(t_0 + \delta, x[t_0 + \delta])$, choose an arbitrary constant control

$$u[t] = u_1 \in U(t_0 + \delta, x[t_0 + \delta]),$$

which determines the motion over the next time interval $[t_0 + \delta, t_0 + 2\delta)$, etc.

The notion of the second player's approximation strategy is introduced in similar fashion. The sets of approximation strategies of the first and second players will be denoted by the symbols $\mathbf{U}_3$ and $\mathbf{V}_3$, respectively. Thus, the set $\mathbf{U}_3$ contains all the possible ways of forming the piecewise-constant controls of the first player; the latter are defined by the functions $U = U(t, x)$ described above and correspond to all possible values of $\delta > 0$.

The above classes of player strategies enable us to define the following types of absorption sets. The first of these, namely the program absorption set $W_1(\tau, \vartheta)$ is defined as the set of points $w \in E_k$ for which the following condition is fulfilled: for any second-player program control $v(\cdot) \in \mathbf{V}_1$ there exists a first-player control $u(\cdot) \in \mathbf{U}_1$ such that the pair of controls $u(t)$, $v(t)$ $(\tau \leqslant t \leqslant \vartheta)$ takes system (1.1) from the position $x(\tau) = w$ to the state $x(\vartheta) \in M$.

The set $W_2^*(\tau, \vartheta)$, i.e. the set of points w, can be defined in similar fashion: for any second-player control $v(\cdot) \in \mathbf{V}_2^*$ there exists a first-player control $u(\cdot) \in \mathbf{U}_1$ such that the pair of controls $u(t)$, $v(t, x)$ takes system (1.1) from the position

$x(\tau) = w$ to the state $x(\vartheta) \in M$.

In defining the absorption sets $W_2(\tau, \vartheta)$ and $W_3(\tau, \vartheta)$ we must take account of the nonuniqueness of the solutions of system (1.1) generated by strategies from the classes $\mathbf{V}_2$ and $\mathbf{V}_3$. In this connection the sets $W_2(\tau, \vartheta)$ and $W_3(\tau, \vartheta)$ can be defined as follows. The set $W_3(\tau, \vartheta)$ $(W_2(\tau, \vartheta))$ is the set of points $w \in E_k$ for which the following condition is fulfilled: for any second-player approximation strategy (for any strategy $v(\cdot) \in \mathbf{V}_2$) there exists a first-player program control $u(\cdot) \in \mathbf{U}_1$ such that the solutions of system (1.1) generated by the first-player program control $u = u(t)$ and by the second-player approximation strategy (the strategy $v(\cdot) \in \mathbf{V}_2$) there exists a solution $x[t]$ which satisfies the conditions $x[\tau] = w$, $x[\vartheta] \in M$.

Thus, the sets $W_1(\tau, \vartheta)$, $W_2(\tau, \vartheta)$, $W_2^*(\tau, \vartheta)$, $W_3(\tau, \vartheta)$ are the sets of initial states $x[\tau] = w$ from which the second player cannot ensure his evading a hit in the set M at the instant ϑ by choosing his strategy from the classes $\mathbf{V}_1$, $\mathbf{V}_2$, $\mathbf{V}_2^*$, $\mathbf{V}_3$, respectively. In similar fashion we can define the absorption sets $S_1(\tau, \vartheta)$, $S_2(\tau, \vartheta)$, $S_2^*(\tau, \vartheta)$ and $S_3(\tau, \vartheta)$ consisting of all the initial positions $x[\tau] = w$ which the second player cannot use to ensure his evading a hit in M for all $t \in [\tau, \vartheta]$ by choosing any strategy from the corresponding class.

Let us describe briefly some of the absorption sets which we have constructed. We know that the system of sets $W_3(\tau, \vartheta)$ $(t_0 \leqslant \tau \leqslant \vartheta)$ is strongly u-stable, and that the system of sets $S_3(\tau, \vartheta)$ $(t_0 \leqslant \tau \leqslant \vartheta)$ is u-stable [7]. This means that given fulfillment of the condition $x_0 \in W_3(t_0, \vartheta)$ (or $x_0 \in S_3(t_0, \vartheta)$) a first-player approximation strategy extremal in the system $W_3(\tau, \vartheta)$ $((S_3(\tau, \vartheta))$ guarantees convergence of system (1.1) to the set M not later than at the instant ϑ; moreover, the inclusion $x_0 \in S_3(t_0, \vartheta)$ is the necessary and sufficient condition whose fulfillment means that the homing game can be concluded by the instant ϑ [7]. Thus, solution of the differential game in question reduces to the construction of the absorption set $S_3(\tau, \vartheta)$ or $W_3(\tau, \vartheta)$. (We note that the set $W_3(\tau, \vartheta)$ coincides to within a nonsingular linear transformation with the alternated integral constructed in [3]).

However, effective construction of these sets is generally a very difficult problem. It is therefore of interest to determine the conditions under which the set $W_3(\tau, \vartheta)$ or $S_3(\tau, \vartheta)$ coincides, let us say, with the set $W_1(\tau, \vartheta)$; construction of this set is a much simpler matter. These matters are dealt with in [8, 9]. The analysis of various absorption sets and the relationships between them also serves to clarify some problems of the structure of differential games. For example, using the equation $W_1(\tau, \vartheta) = W_2(\tau, \vartheta) = W_2^*(\tau, \vartheta)$ which we shall prove in the present paper, we can show that an evasion strategy generally cannot be approximated by continuous game position vector functions, i.e. by strategies from the class $\mathbf{V}_2$ or $\mathbf{V}_2^*$. An example illustrating this fact is given at the end of the present paper.

2. Let us consider the sets $W_1(\tau, \vartheta)$, $W_2(\tau, \vartheta)$ and $W_2^*(\tau, \vartheta)$. We assert that these sets coincide. This assertion has the following meaning. If some position $\{\tau, w\}$ is such that for any program control $v(\cdot) \in \mathbf{V}_1$ there exists a program control $u(\cdot) \in \mathbf{U}_1$ which brings system (1.1) to M at the instant ϑ, then the second player cannot ensure his evading a hit in the set M at the instant ϑ if he chooses his controls from the class $\mathbf{V}_2$ or from the class $\mathbf{V}_2^*$. On the other hand, if the position $\{\tau, w\}$ is such that, for example, the class $\mathbf{V}_2$ contains some strategy $v(t, x)$ which insures the second player against the arrival of system (1.1) in M at the instant ϑ, then the classes of controls $\mathbf{V}_1$ and $\mathbf{V}_2^*$ also contain second-player strategies which ensure his evasion of a hit at M at the instant ϑ for any control chosen by the first player.

Now let us prove the following theorem.

Theorem 2.1. The sets $W_1(\tau, \vartheta)$, $W_2(\tau, \vartheta)$ and $W_2^*(\tau, \vartheta)$ coincide, i.e.

$$W_1(\tau, \vartheta) = W_2(\tau, \vartheta) = W_2^*(\tau, \vartheta) \quad (-\infty < \tau \leqslant \vartheta < \infty) \tag{2.1}$$

Proof. We begin by proving the equation

$$W_1(\tau, \vartheta) = W_2^*(\tau, \vartheta) \tag{2.2}$$

The inclusion $W_2^*(\tau, \vartheta) \subset W_1(\tau, \vartheta)$ is self-evident (since the set of controls $\mathbf{V}_2^*$ contains the set of controls $\mathbf{V}_1$). It remains for us to show that the inclusion $W_2^*(\tau, \vartheta) \supset W_1(\tau, \vartheta)$ holds. Let us suppose that this is not the case. Let there exist τ and ϑ $(\vartheta \geqslant \tau)$ such that

$$W_2^*(\tau, \vartheta) \not\supset W_1(\tau, \vartheta) \tag{2.3}$$

i.e. that there exists a point w_* for which the relations

$$w_* \in W_1(\tau, \vartheta), \qquad w_* \notin W_2^*(\tau, \vartheta) \tag{2.4}$$

are valid simultaneously.

The second condition of (2.4) has the following meaning: there exists a function $v^*(\cdot) \in \mathbf{V}_2^*$ such that for any program control $u(\cdot) \in \mathbf{U}_1$ the solution

$$x(t) \quad (x[\tau] = w_*, \ \tau \leqslant t \leqslant \vartheta)$$

generated by a pair of controls $u = u[t]$, $v = v^*(t, x)$ satisfies the condition $x[\vartheta] \notin M$. (We note once again that the motion of system (1.1) in this case is unique by virtue of the Carátheodory theorem [10]).

We shall consider the sets $\mathbf{V}_1$ and $\mathbf{U}_1$ as some sets in the space $L_2[\tau, \vartheta]$. We note that these sets are convex, bounded, and weakly compact in $L_2[\tau, \vartheta]$.

Now let us construct the following mapping of the set $\mathbf{U}_1$ into itself: each element $u(\cdot) \in \mathbf{U}_1$ is associated with some set $\Phi_u \subset \mathbf{U}_1$ (the elements of the set Φ_u are some program controls $\varphi(\cdot) \in \mathbf{U}_1$). We note that all the program controls are being considered in the interval $[\tau, \vartheta]$. The set Φ_u is defined as follows.

Let $u(\cdot) \in \mathbf{U}_1$ be an arbitrary element from the set of first-player program controls. By $x_u[t]$ $(\tau \leqslant t \leqslant \vartheta)$ we denote the motion of system (1.1) from the initial state $x_u[\tau] = w_*$ generated by this program control $u = u(t)$ and by the second player's control $v = v^*(t, x)$. We denote the realization of the second-player control $v = v^*(t, x)$ computed along the motion $x = x_u[t]$ by $v_u^*[t]$, i.e.

$$v_u^*[t] = v^*(t, x_u[t]) \quad (\tau \leqslant t \leqslant \vartheta).$$

It is not difficult to verify that $v_u^*[\cdot] \in \mathbf{V}_1$. We now define the set Φ_u as the set of all first-player program controls $\varphi(\cdot) \in \mathbf{U}_1$ which, on being paired with the program control $v = v_u^*[t]$ take system (1.1) from the state $x[\tau] = w_*$ to the position $x[\vartheta] \in M$. We note that since the point $w_* \in W_1(\tau, \vartheta)$ (see (2.4)), it follows that the set Φ_u is nonempty for all $u(\cdot) \in \mathbf{U}_1$.

From the construction of the set Φ_u and from the properties of the control $v = v^*(t, x)$ we see that the condition

$$u(\cdot) \notin \Phi_u \tag{2.5}$$

must be fulfilled for all $u(\cdot) \in \mathbf{U}_1$.

Thus, in order to arrive at a contradiction of conditions (2.4) we need merely prove the existence of a control $u^\circ(\cdot) \in \mathrm{U}_1$ for which

$$u^\circ(\cdot) \in \Phi_{u^\circ} \tag{2.6}$$

We can show that there exists a control $u^\circ(\cdot) \in \mathrm{U}_1$ which satisfies (2.6) and thereby prove Eq. (2.2).

To demonstrate the existence of such a control $u^\circ(\cdot)$ we make use of the Bohnenblust-Karling fixed-point theorem ([11], p. 496).

Let us verify the fulfillment of the conditions of this theorem. First, we recall that the controls $u(\cdot) \in \mathrm{U}_1$ are regarded as elements of the space $L_2[\tau, \vartheta]$, and that the set U_1 is convex, weakly compact in $L_2[\tau, \vartheta]$, and bounded. We can show that the set Φ_u is convex for all $u(\cdot) \in \mathrm{U}_1$. Let $\varphi_1(\cdot) \in \Phi_u$, $\varphi_2(\cdot) \in \Phi_u$. Then the control

$$\varphi_\lambda(t) = \lambda\varphi_1(t) + (1-\lambda)\,\varphi_2(t),\quad 0 \leqslant \lambda \leqslant 1$$

also belongs to the set Φ_u, i.e. the pair of controls $\varphi_\lambda(t)$, $v_u^*[t]$ $(\tau \leqslant t \leqslant \vartheta)$ brings system (1.1) from $x[\tau] = w_*$ to M at the instant ϑ. In fact, in order to verify this we need merely write out the Cauchy formula

$$x_i[\vartheta] = X(\vartheta, \tau)\, w_* + \int_\tau^\vartheta X(\vartheta, t)\, B(t)\, \varphi_i(t)\, dt - \int_\tau^\vartheta X(\vartheta, t)\, C(t)\, v_u^*[t]\, dt, \quad i=1, 2$$

where $X(t, \tau)$ is the fundamental matrix of the homogeneous system

$$dx/dt = A(t)x$$

Recalling that M is convex and that $x_i[\vartheta] \in M$, we obtain the inclusion

$$x_\lambda[\vartheta] = \lambda x_1[\vartheta] + (1-\lambda)x_2[\vartheta] \in M,$$

in other words, the control $\varphi_\lambda(t)$ also belongs to the set Φ_u.

Now let us verify the fulfillment of the following condition: if the sequence $u_n(\cdot) \in \mathrm{U}_1$ converges weakly to $u^*(\cdot)$ and if the sequence $\varphi_n(\cdot)$ converges weakly to $\varphi^*(\cdot)$, where $\varphi_n(\cdot) \in \Phi_{u_n}$, $n = 1, 2, \ldots$, then $\varphi^*(\cdot) \in \Phi_{u^*}$. (The semicontinuity condition.)

For simplicity, in proving discontinuity we shall denote the set Φ_{u_n} by Φ_n and $v^*_{u_n}[\cdot]$, $x_{u_n}[\cdot]$ by $v_n^*[\cdot]$, $x_n[\cdot]$.

First we note that $x_n[\cdot]$ is a sequence of uniformly bounded and equicontinuous functions, so that by the Arzelà theorem ([12], p. 236) we can isolate a uniformly convergent subsequence $x_{n_i}[\cdot]$, $\lim_{i\to\infty} x_{n_i}[\cdot] = x^*[\cdot]$, where, clearly, $x^*[\tau] = w_*$. By virtue of the continuity of the vector function $v^*(t, x)$ in x and the uniform convergence of the subsequence $x_{n_i}[\cdot]$ to $x^*[\cdot]$, the realizations of the control $v^*(t, x)$ computed along the trajectories $x_{n_i}[\cdot]$ converge to $v^*(t, x^*[t])$ at every point $t \in [\tau, \vartheta]$.

Now let us express the motions $x_{n_i}[\cdot]$ in the form

$$x_{n_i}[t] = \int_\tau^t A(\xi)\, x_{n_i}[\xi]\, d\xi + \int_\tau^t B(\xi)\, u_{n_i}(\xi)\, d\xi - \int_\tau^t C(\xi)\, v^*(\xi, x_{n_i}[\xi])\, d\xi + w_* \tag{2.7}$$

In this equation we can also take the limit for a fixed $t \in [\tau, \vartheta]$ as $i \to \infty$. On the left we obtain $x^*[t]$. The first two terms in the right side of (2.7) have a limit by virtue of the uniform convergence of $x_{n_i}[\cdot]$ to $x^*[\cdot]$ and the weak convergence of $u_{n_i}(\cdot)$ to $u^*[\cdot]$. This means that the third term in the right side also has a limit; moreover, by condition (1.3), $v^*(\xi, x) \in Q$ and $v^*_{n_i}[\xi]$ converges to $v^*(\xi, x^*[\xi])$ at every point $\xi \in [\tau, \vartheta]$; this means that the limit of the integrand of the third term can be taken. We obtain

$$x^*[t] = \int\limits_{\tau} (A(\xi)\, x^*[\xi] + B(\xi)\, u^*(\xi) - C(\xi)\, v^*(\xi, x^*[\xi]))\, d\xi + w_*$$

The integrand is summable over $[\tau, \vartheta]$, so that almost everywhere in $[\tau, \vartheta]$ we have

$$\frac{dx^*[t]}{dt} = A(t)\, x^*[t] + B(t)\, u^*(t) - C(t)\, v^*(t, x^*[t]) \tag{2.8}$$

$$x^*[\tau] = w_*$$

i.e. $x^*[t]$ is the unique (by virtue of the Caratheodory theorem) solution of system (1.1) corresponding to the controls $u = u^*[t]$ and $v = v^*(t, x)$; the control $v_{u^*}^*[t] = v^*(t, x^*[t])$ is realized in this case. Now let us show that $\varphi^*(\cdot) \in \Phi_{u^*}$. The condition $\varphi_{n_i}(\cdot) \in \Phi_{n_i}$ means that

$$X(\vartheta, \tau)\, w_* + \int\limits_{\tau}^{\vartheta} X(\vartheta, \xi)\, B(\xi)\, \varphi_{n_i}(\xi)\, d\xi - \int\limits_{\tau}^{\vartheta} X(\vartheta, \xi)\, C(\xi)\, v_{n_i}^*[\xi]\, d\xi = y_{n_i} \in M \tag{2.9}$$

As $i \to \infty$ the sequence y_{n_i} converges to some point, which belongs to this set by virtue of the closure of the set M. Hence, taking the limit in Eq. (2.9), we obtain

$$X(\vartheta, \tau)\, w_* + \int\limits_{\tau}^{\vartheta} X(\vartheta, \xi)\, B(\xi)\, \varphi^*(\xi)\, d\xi - \int\limits_{\tau}^{\vartheta} X(\vartheta, \xi)\, C(\xi)\, v_{u^*}^*[\xi]\, d\xi = y_* \in M \tag{2.10}$$

(The possibility of taking the limit in the first integral follows from the weak convergence of the sequence $\varphi_n(\cdot)$; taking of the limit in the second integral can be justified in the same way as in (2.7) above). Relation (2.10) means that the control $\varphi^*(\cdot)$ paired with the control $v_{u^*}^*[\cdot]$ takes system (1.1) from the position $x[\tau] = w_*$ to the state $x[\vartheta] \in M$; hence, $\varphi^*(\cdot) \in \Phi_{u^*}$. The condition of semicontinuity has been proved.

The last condition of the fixed-point theorem ([11], p. 496) follows from the weak compactness of the set U_1 and from its boundedness in $L_2[\tau, \vartheta]$. Thus, all the conditions of the fixed-point theorem have been satisfied, so that Eq. (2.2) has been proved.

Let us take note of the principal arguments used to prove the coincidence of the set $W_2(\tau, \vartheta)$ with the sets $W_1(\tau, \vartheta)$ and $W_2^*(\tau, \vartheta)$. Let $v(t, x)$ be an arbitrary control from the set V_2. We know that every vector function $v = v(t, x)$ can be expressed as the limit of some vector functions $v = v_n(t, x)$, each of which belongs to the class V_2^*; moreover, the convergence of $v_n(t, x)$ to $v(t, x)$ is uniform on every bounded set $D \subset E_{k+1}$. Making use of this fact, we can prove that in the case where

$$x[\tau] = w_* \in W_1(\tau, \vartheta) = W_2^*(\tau, \vartheta)$$

the solutions generated by the control $v = v(t, x)$ and by some control $u(\cdot) \in U_1$ there exists a solution which satisfies the conditions $x[\tau] = w_*$, $x[\vartheta] \in M$, i.e that $w_* \in W_2(\tau, \vartheta)$, and therefore $W_2(\tau, \vartheta) \supset W_1(\tau, \vartheta) = W_2^*(\tau, \vartheta)$ The inclusion $W_2(\tau, \vartheta) \subset W_1(\tau, \vartheta)$ follows from the possibility of approximating the measurable program controls $v(\cdot) \in V_1$ by continuous program controls. We can therefore justify the validity of the equation $W_1(\tau, \vartheta) = W_2(\tau, \vartheta) = W_2^*(\tau, \vartheta)$ and of Theorem 2.1.

3. In conclusion let us consider an example of a pursuit game in which the evasion strategy cannot be approximated by strategies from the classes V_2 and V_2^*. Let the motion of the system be described by the equations

$$dy/dt = z - v, \qquad dz/dt = u \tag{3.1}$$

Here y, z are the vectors of the Euclidean space E_2 ; the players' controls u and v are subject to the restrictions $\|u\| \leqslant \mu$, $\|v\| \leqslant \nu$, where $\mu > 0$, $\nu > 0$ are some constant numbers; the set M is the hyperplane $y = 0$. We note that Eqs (3.1) describe the process of pursuit of the inertialess point $m^{(2)}$ by the material point of unit mass $m^{(1)}$ controlled by the force u; the components of the vectors y are the differences between the corresponding coordinates of the points $m^{(1)}$ and $m^{(2)}$; the vector z is the velocity of the material point $m^{(1)}$. We know this problem does, in fact, contain an evasion strategy, i.e. that there exists some method of forming the control v which enables the second player to evade encounter with the point $m^{(1)}$.

The authors of [6] proved the existence of such a strategy in the class of second-player controls which are formed with the aid of immediate information on the choice of the first-player control $u[t]$ at each instant $t \geqslant t_0$.

It can be shown that in this class evasion of encounter over an arbitrary large time interval is ensured by the approximation strategy defined by the function $V = V_*(t, y, z)$. This function $V = V_*(t, y, z)$ associates the position $p = \{t, y, z\}$ with the set of vectors v_* satisfying the conditions

$$(y, v_*) \leqslant 0, \ (z, v_*) = 0, \quad \|v_*\| = \nu \tag{3.2}$$

Here the symbol (y, v) is the scalar product of the vectors y and v. Thus, the vector v_* is orthogonal to the vector z, and the projection of the vector v_* onto the vector y is nonpositive. Let us note the following fact. The function $V = V_*(t, y, z)$ is not a single-valued continuous function of the position $p = \{t, y, z\}$; if the vectors y and z are noncolinear, then for $y \neq 0$ the set $V_*(t, y, z)$ contains two vectors; for $z = 0$ the set $V_*(t, y, z)$ consists of all possible vectors v_*, $\| v_* \| = \nu$ $(y, v_*) \leqslant 0$.

Thus, in the example under consideration there exists some method of forming the second-player control which guarantees that the latter will be able to evade the pursuing player. Let us show that the second player cannot evade encounter by means of any strategy from the class V_2 or V_2^*. In fact, it is not difficult to verify that in this case for any initial game position $p_0 = \{t_0, x_0\} = \{t_0, y_0, z_0\}$ there exists a parameter value $\vartheta^\circ < \infty$ such that the inclusion

$$x_0 \in W_1(t_0, \vartheta^\circ) \tag{3.3}$$

holds.

But then, by virtue of Theorem 2.1, we also have the inclusions

$$x_0 \in W_2(t_0, \vartheta^\circ), \qquad x_0 \in W_2^*(t_0, \vartheta^\circ) \tag{3.4}$$

which, by the definitions of the absorption sets $W_2(\tau, \vartheta)$ and $W_2^*(\tau, \vartheta)$, means that the pursued player cannot evade encounter by choosing any strategy $v(\cdot) \in V_2$ or $v(\cdot) \in V_2^*$. Moreover, encounter occurs not later than at the program absorption instant $t = \vartheta^\circ$. Considerations similar to those used to prove Theorem 2.1 can also be used to show that the above situation continues to hold even if the second player chooses any control $v[t] = v(t, x[t], u[t])$, where $v(t, x, u)$ is some continuous vector function.

Thus, the optimal or near-optimal second-player strategies cannot be sought in the class of continuous game position vector functions. We note, however, that in some cases the solution of an evasion problem can be approximated by means of the continuous strategies $v(t, x)$. This class of evasion problems includes those differential games which conform to the conditions formulated in [5].

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ON A CERTAIN CONVERGENCE GAME

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A game problem on the convergence of controlled objects by the instant $t = \vartheta$ is considered in a fixed time interval $[t_0, \vartheta]$. It is assumed that the pursuing object is an inertial point and that the pursued object is inertialess. The problem of constructing the pursuer's optimal minimax strategy is considered. This strategy ensures the minimax of the distance between the objects at a given instant. It is proved that the mixed strategy of special form (derived in [3]) which operates within the framework of the mathematical apparatus of differential equations in contingencies is such a strategy.

1. Let us consider a differential game involving the two objects $m^{(1)}$ and $m^{(2)}$ moving in the horizontal plane q_1q_2. The motion of the pursuing object $m^{(1)}(y_1, y_2)$ controlled by the first player is described by the system of equations

$$y_1^{\cdot} = y_3, \qquad y_2^{\cdot} = y_4, \qquad y_3^{\cdot} = u_3, \qquad y_4^{\cdot} = u_4 \tag{1.1}$$

where the control vector $u = u^*(u_3, u_4)$ satisfies the inequality

$$(u_3^2[t] + u_4^2[t])^{1/2} \leqslant \mu \tag{1.2}$$